Analysis of Grid Search Algorithm

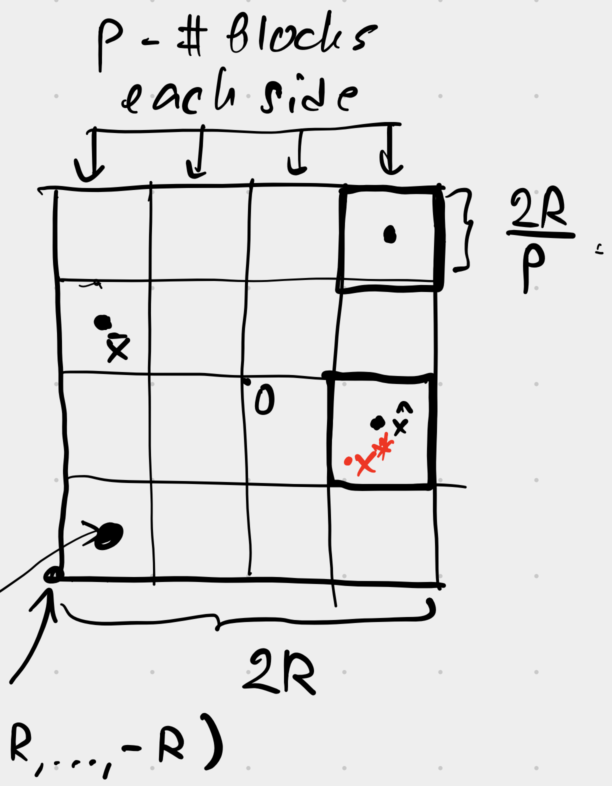
Problem: $\min_{x \in B} f(x)$

- $B = \{ x \in \mathbb{R}^n : \|x\|_\infty \le R \}$
- f is Lipschitz continuous:

$$|f(y) - f(x)| \le L\|y - x\|_\infty \quad \forall y, x \in B$$

Parameters of Problem class:
1. Dimension $n \ge 1$
2. $R > 0$
3. $L > 0$

Accuracy condition: $f(\bar{x}) - f^* \le \varepsilon$



Algorithm (Grid Search)

1. Choose $p \ge 1$
2. Generate p^n points

$$x_{(t_1,\dots,t_n)} = \left[-\frac{(p-1)}{p}R + \frac{2R}{p}t_1, \dots, -\frac{(p-1)}{p}R + \frac{2R}{p}t_n\right]$$

For all $0 \le t_i \le p-1$

3. Find the point $\bar{x}$ among all generated points with smallest funct. value. Return $\bar{x}$.

$$(-R,\dots,-R) + \left(\frac{R}{p},\dots,\frac{R}{p}\right) = \left(-\frac{(p-1)}{p}R, \dots, -\frac{(p-1)}{p}R\right)$$

Theorem $f(\bar{x}) - f^* \le \frac{2LR}{p}$.

Proof. $\exists$ small box $B_* \ni x_*$.

Denote its center by $\hat{x}$.

$$f(\bar{x}) - f^* = f(\bar{x}) - f(x^*) \overset{\text{Step 3}}{\le} f(\hat{x}) - f(x^*) \le L\|\hat{x} - x^*\|_\infty \le \frac{2LR}{p}. \quad \square$$

Complexity? $f(\bar{x}) - f^* \leq \varepsilon$

$f(\bar{x}) - f^* \leq \frac{2LR}{p} \overset{?}{\leq} \varepsilon \Rightarrow$ It's sufficient to set

$$p = \left\lfloor \frac{2LR}{\varepsilon} \right\rfloor + 1$$

The number of oracle calls:

$$\boxed{K = p^n = \left(\left\lfloor \frac{2LR}{\varepsilon} \right\rfloor + 1 \right)^n}$$

Is it good? - Not really, e.g. $L = R = 1$

$$O\left(\left(\frac{1}{\varepsilon} \right)^n \right)$$

$$\varepsilon = 10^{-2} \quad n \geq 50 \Rightarrow O(10^{100})$$

Better algorithm? - No.

Lower Bound

Theorem Complexity of any zero-th-order method for our problem class is at least $\left\lfloor \frac{RL}{\varepsilon} \right\rfloor^n$.

Upper Bound				Lower Bound
$\left(\left\lfloor \frac{2LR}{\varepsilon} \right\rfloor + 1 \right)^n$	$\geq$	K	$\geq$	$\left\lfloor \frac{LR}{\varepsilon} \right\rfloor^n$

$\Rightarrow$ The grid search is optimal.

Resisting oracle $\boxed{\text{Return } 0.}$