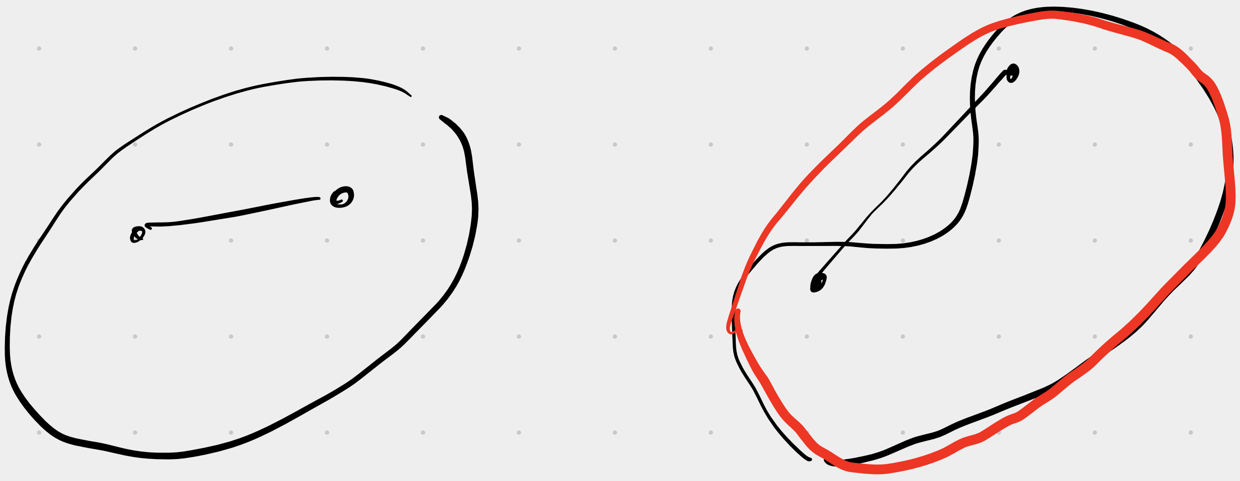
Convex Sets

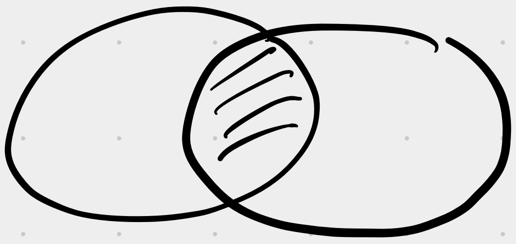
$Q \subseteq \mathbb{R}^n$ convex, $\forall x, y \in Q$:

$$\lambda x + (1-\lambda) y \in Q \qquad \lambda \in [0, 1]$$



Basic Properties

1. Intersection



Q_α - convex, α

$$Q = \bigcap_\alpha Q_\alpha \quad \text{- convex}$$

2. Convex Hull

$$\mathrm{conv}(Q) = \bigcap \{ Q_\alpha \subseteq \mathbb{R}^n \text{ - convex, contains } Q \}$$

3. Affine image of convex set.

$$A(x) = Ax + b$$

$$A(Q) = \{ A(x) \mid x \in Q \} \text{ - convex,}$$

Q - convex

Examples

1. Hyperplane and half-space

$Q = \{x \in \mathbb{R}^n : \langle a, x\rangle = b\}$ $\qquad$ $Q = \{x \in \mathbb{R}^n : \langle a, x\rangle \leq b\}$

2. Affine space:

$Q = \{x \in \mathbb{R}^n ; Ax = b\}$

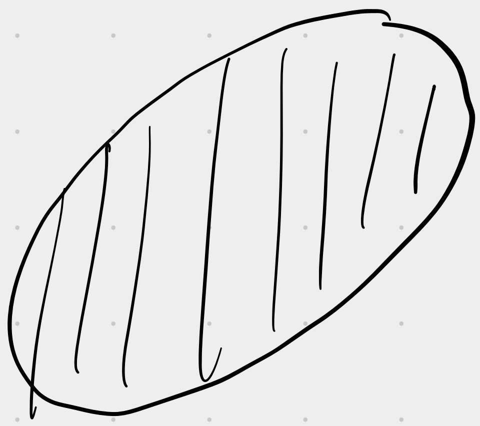
Polyhedron

$Q = \{x \in \mathbb{R}^n : Ax \leq b\}$

3. Ellipsoid

$$Q = \{x \in \mathbb{R}^n \mid \langle A(x - x_0), x - x_0\rangle \leq 1\}$$

$A = A^T \succ 0, \qquad x_0 \in \mathbb{R}^n$



$$Q = \{Bu + x_0 \mid u \in \mathbb{R}^n \ \langle u, u\rangle \leq 1\}$$

$B = A^{-1/2}$.

6. Cone of positive semidefinite matrices:

$Q = \{X \in S^n \mid X \succcurlyeq 0\}$ - convex cone

$X \succcurlyeq 0, \ \alpha > 0 \quad \Rightarrow \quad \alpha X \succcurlyeq 0 \qquad \underline{\alpha X \in Q}$

$X, Y \succcurlyeq 0 \quad \Rightarrow \quad X + Y \succcurlyeq 0 \qquad X + Y \in Q$

Semidefinite Programming:

$$Q = \{ X \in S^n \mid X \succeq 0, \ \langle A_1, X \rangle = b_1, \dots, \langle A_m, X \rangle = b_m \}$$

set X - diagonal $\Rightarrow$ LP.

Epigraph of Convex Function

$f : \mathrm{dom} f \to \mathbb{R}$, $\mathrm{dom} f \subseteq \mathbb{R}^n$

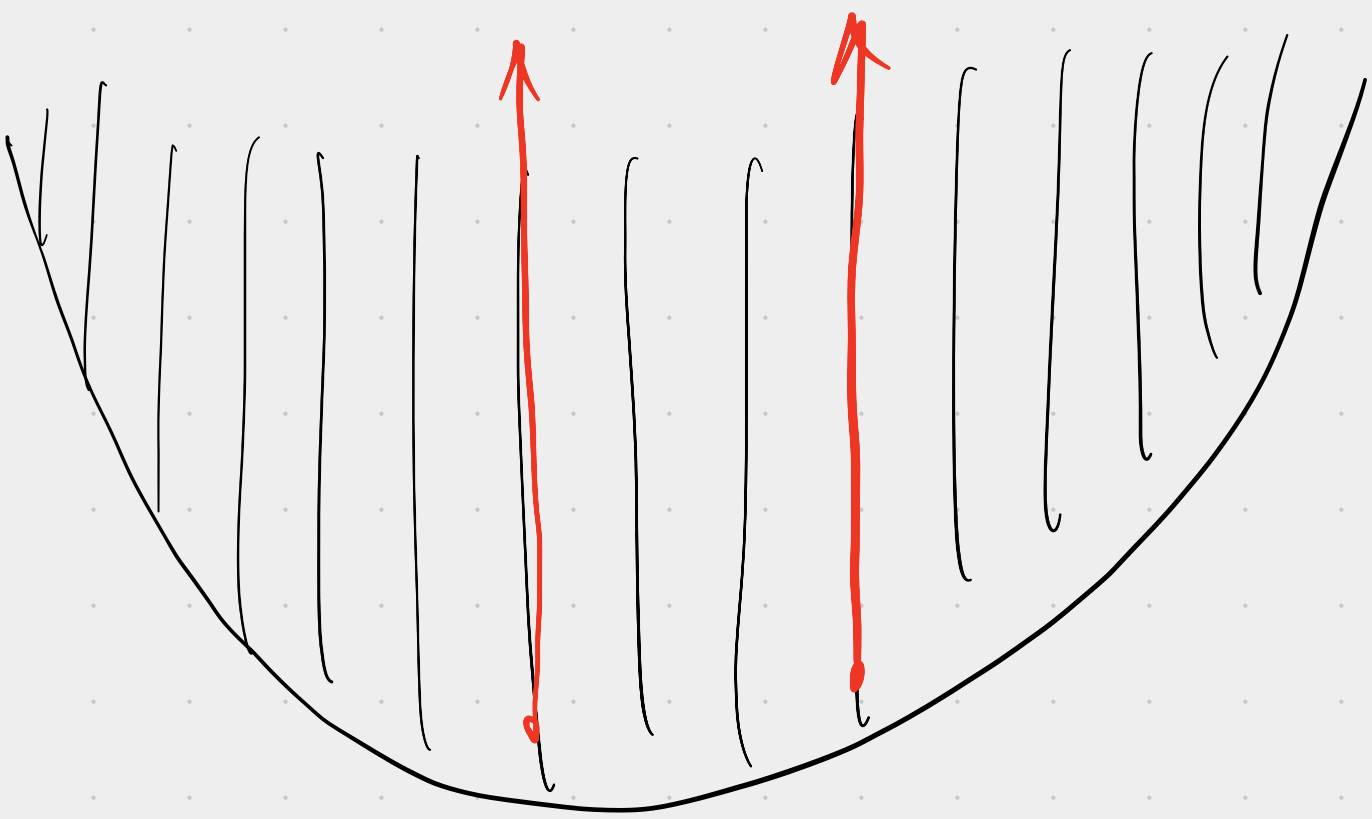
$\forall x, y \in \mathrm{dom} f \quad \forall\ 0 \le \lambda \le 1$:

$$f(\underbrace{\lambda x + (1-\lambda) y}_{\in \mathrm{dom} f}) \le \lambda f(x) + (1-\lambda) f(y).$$

By definition: $\mathrm{dom} f$ is convex

Epigraph $\quad \mathrm{dom} f \subseteq \mathbb{R}^n$

$$\mathrm{epi} f = \{ (x, t) \in \mathrm{dom} f \times \mathbb{R} \mid f(x) \le t \} \subseteq \mathbb{R}^{n+1}$$



Prop. f is convex $\Leftrightarrow$ $\mathrm{epi} f$ is a convex set.

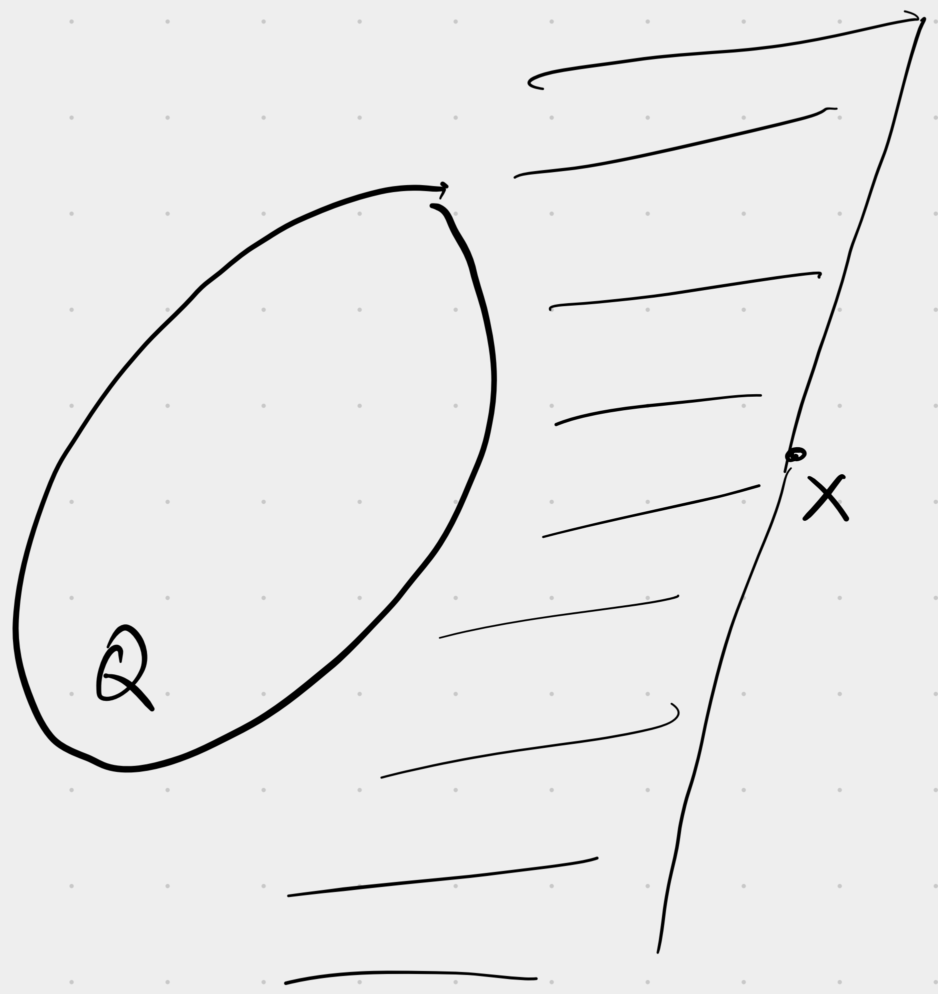
Separation Theorem

<u>Theorem</u> let $Q \subseteq \mathbb{R}^n$, assume $\operatorname{int} Q \neq \emptyset$.

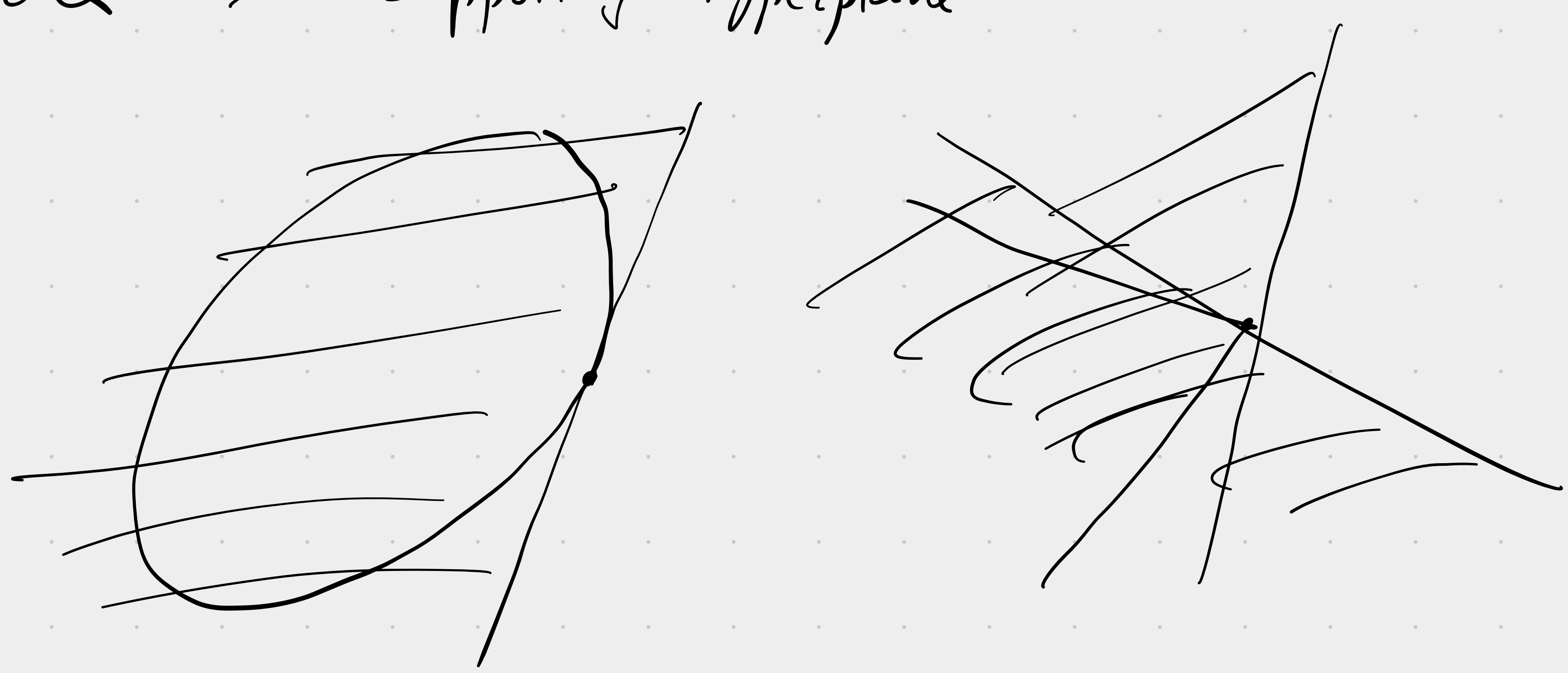
let $x \notin \operatorname{int} Q$. $\Rightarrow$ x can be separated from Q by a linear (nonzero) function

$\exists\, \ell \in \mathbb{R}^n,\ \ell \neq 0$:

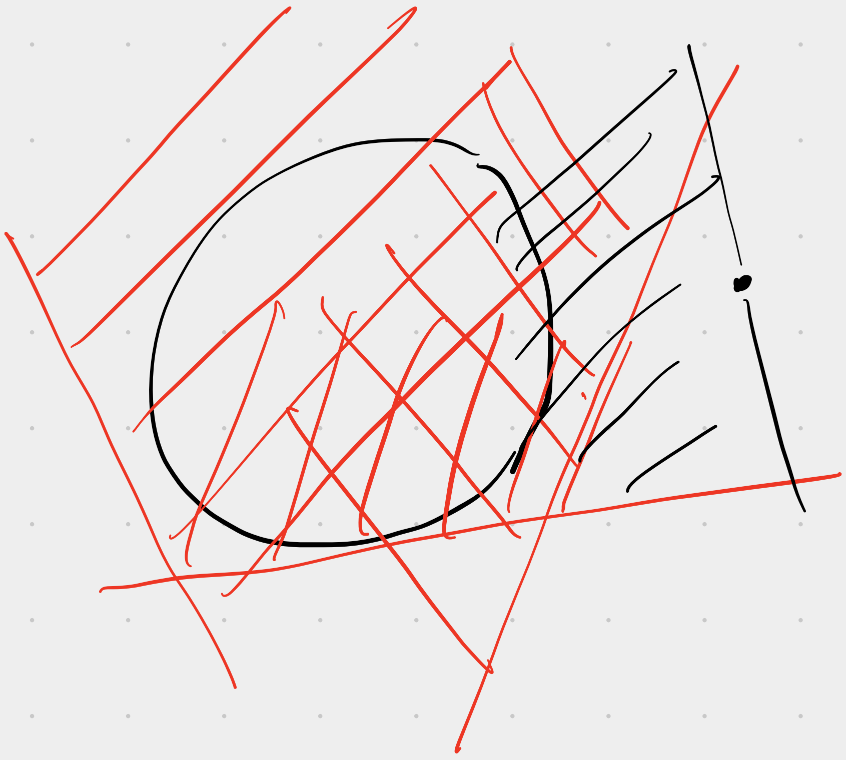
$$\langle \ell, x\rangle \geq \langle \ell, y\rangle, \quad y \in Q.$$



$x \in \partial Q \Rightarrow$ "Supporting hyperplane"



<u>Corollary</u> Any closed (open) convex set $Q \subseteq \mathbb{R}^2$ is equal to the <u>intersection of all</u> closed (open) half spaces containing it.



$$Q \subseteq \bigcap \{\text{half spaces containing } Q\}$$

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Subgradient

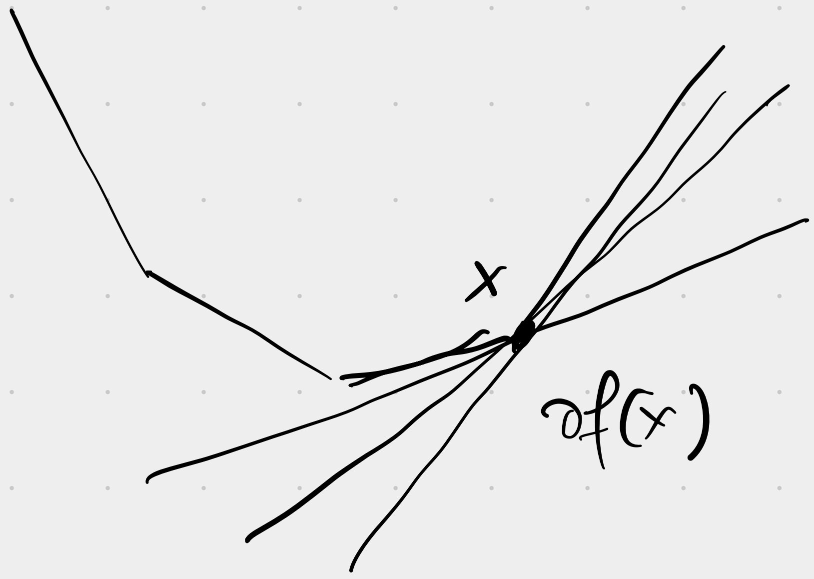
$f: Q \to \mathbb{R}$, $Q \subseteq \mathbb{R}^n$. f is convex.

We say $g \in \mathbb{R}^n$ is subgradient at point $x \in Q$:

$$f(y) \geq f(x) + \langle g, y-x \rangle \quad \forall y \in Q.$$

$\partial f(x)$ = set of all subgradients

subdifferential of f at x.

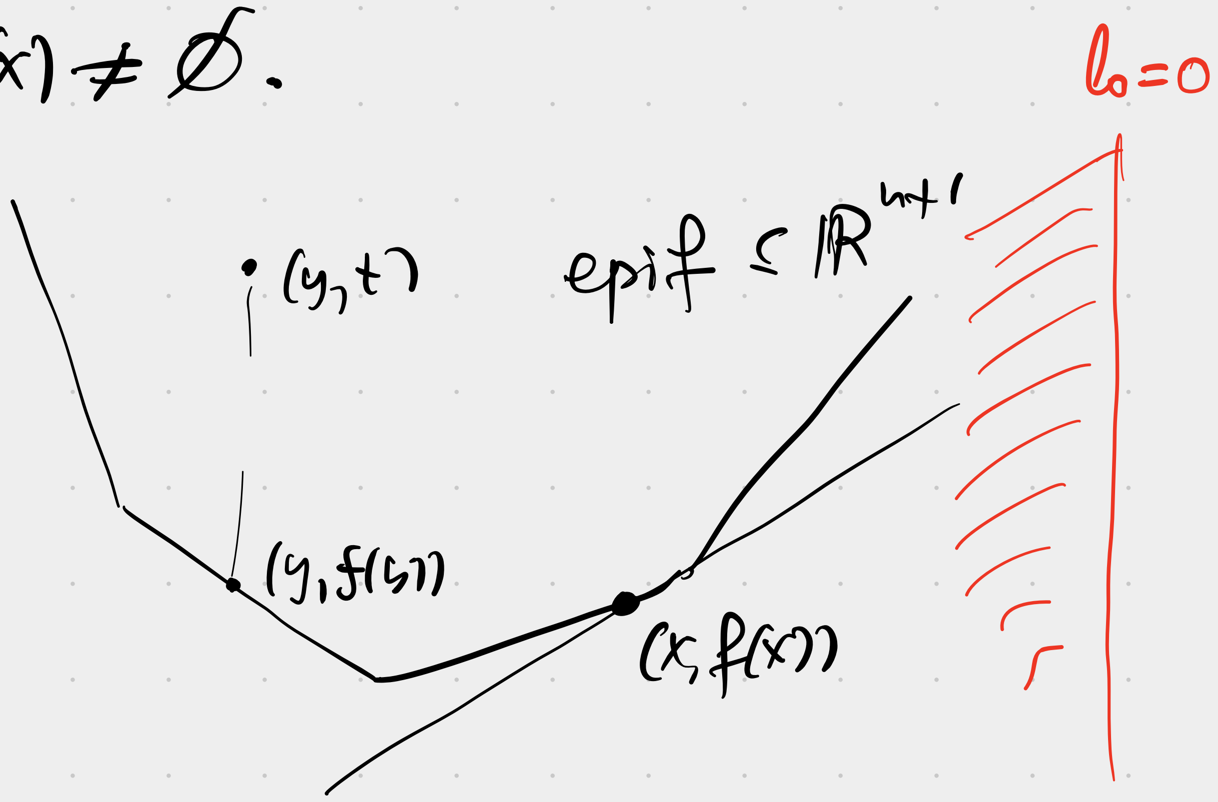


$f: Q \to \mathbb{R}$, Q - convex and <u>open</u>

f - convex and continuous on $Q \subseteq \mathbb{R}^n$.

<u>Theorem</u> $\forall x \in Q \quad \partial f(x) \neq \emptyset$.

<u>Proof</u>



$(x, f(x)) \in \partial \operatorname{epi} f$

$\exists \; \begin{matrix} l_0 \in \mathbb{R} \\ l \in \mathbb{R}^n \end{matrix} : \quad \left\langle \begin{bmatrix} l_0 \\ l \end{bmatrix}, \begin{bmatrix} t - f(x) \\ y - x \end{bmatrix} \right\rangle \geq 0 \quad \begin{matrix} \forall y \in Q \\ t \geq f(y) \end{matrix}$

$$\underline{l_0}\cdot(t-f(x)) + \langle \underline{l, y-x} \rangle \geq 0$$

① $y:=x,\ t>f(x) \Rightarrow l_0\cdot(t-f(x))\geq 0 \Rightarrow \underline{l_0\geq 0}$

② $l_0>0$.

Assume $l_0=0$.

$$\underset{\substack{\cap \\ Q}}{y} := x-\varepsilon\frac{l}{\|l\|},\quad \varepsilon>0 \Rightarrow \langle l, y-x\rangle = -\varepsilon\|l\| < 0$$

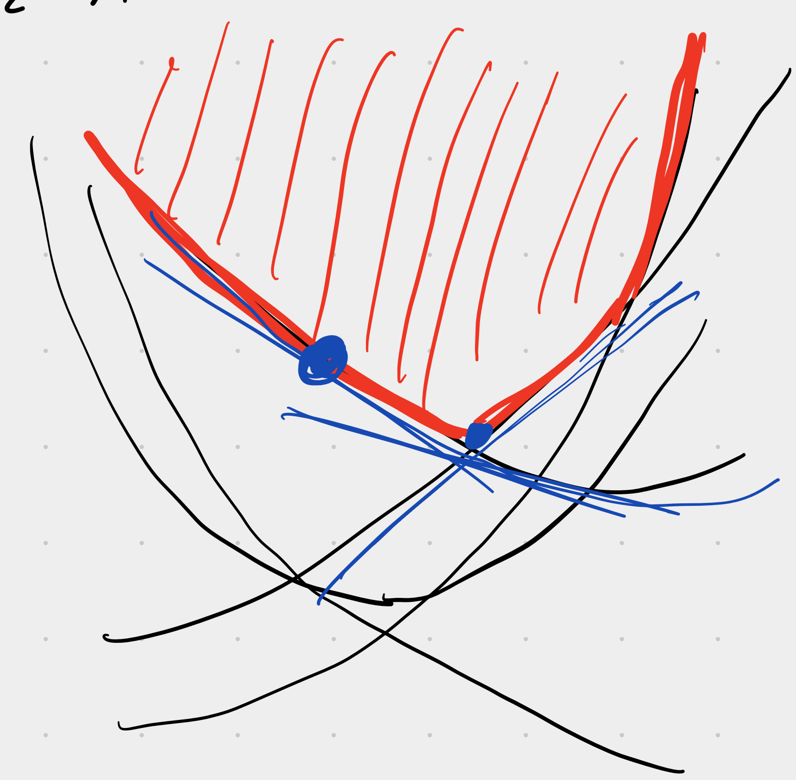
(?!)

Therefore, $t=f(y)$:

$$f(y)\geq f(x)+\langle \frac{l}{l_0}, y-x\rangle \Rightarrow \frac{l}{l_0}\in\partial f(x). \square$$

<u>Properties</u>

- f is differentiable $\Rightarrow \partial f(x)=\{\nabla f(x)\}$
- $f(x)=\alpha f_1(x)+\beta f_2(x),\ \alpha,\beta\geq 0 \Rightarrow$
  $\partial f(x)=\alpha\,\partial f_1(x)+\beta\,\partial f_2(x)$.
- $\underset{\text{convex}}{f(x)} = \max_\alpha f_\alpha(x)$



$$\text{epi} f = \bigcap_\alpha \text{epi} f_\alpha$$

$$\partial f(x) \supseteq \operatorname{conv}\{ \partial f_\alpha(x) \mid f_\alpha(x) = f(x) \}$$

$$f(y) \geq f_\alpha(y) \geq f_\alpha(x) + \langle f_\alpha'(x), y-x \rangle, \qquad f_\alpha'(x) \in \partial f_\alpha(x)$$

$$= f(x) + \langle f_\alpha'(x), y-x \rangle.$$

$$\Rightarrow f_\alpha'(x) \in \partial f(x).$$

Basically: $f(x) = \max_{1 \leq i \leq m} f_i(x) \Rightarrow \partial f(x) = \operatorname{Conv}\{ \partial f_i(x) \mid f_i(x) = f(x) \}$

$X \in S^n$

$$f(X) = \lambda_{max}(X) = \max_{\|u\|=1} \langle Xu, u \rangle = \max_{\|u\|=1} \underbrace{\operatorname{Tr}(Xuu^T)}_{\text{convex}}$$

$$f_u(X) = \operatorname{Tr}(Xuu^T)$$

$$\nabla f_u(X) = uu^T$$

$$\partial f_u(X) = \{uu^T\}$$

$$\partial f(X) \supseteq \operatorname{Conv}\{ uu^T : Xu = \lambda_{max}(X) u \}$$

$=$

First-order oracle

$X \mapsto (f(x), f'(x))$, $f'(x) \in \partial f(x)$.

any subgradient.