Problem

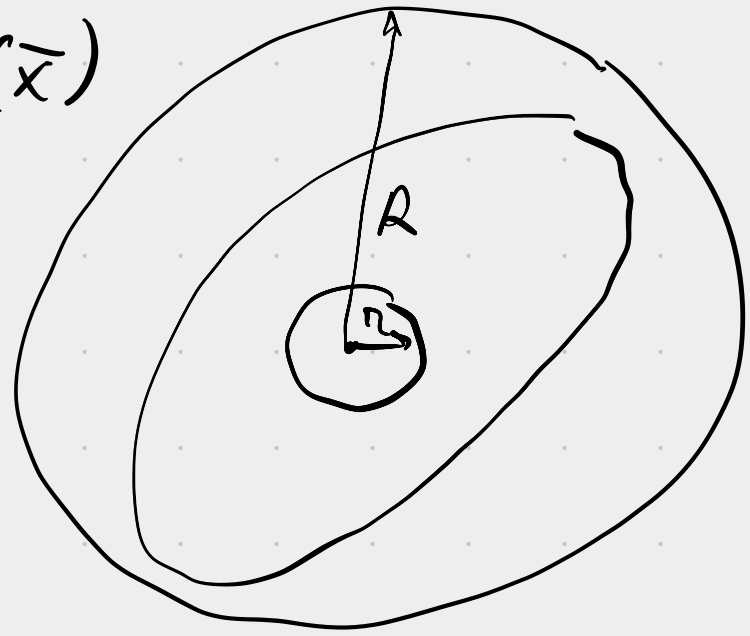
$$\min_{x \in Q} f(x)$$

$f: Q \to \mathbb{R}$ - convex, $Q \subseteq \mathbb{R}^n$ - convex

- Set Q is bounded and it has nonempty interior.

$\exists\, \bar{x} \in Q \quad 0 < r \le R < +\infty$:

$$B_r(\bar{x}) \subseteq Q \subseteq B_R(\bar{x})$$



where $B_\alpha(\bar{x}) = \{ \|y - \bar{x}\|_2 \le \alpha \}$

$\frac{R}{r} \ge 1$ called asphericity of Q.

- $f: Q \to \mathbb{R}$ has bounded variation:

$$V = \max_{x \in Q} f(x) - \min_{x \in Q} f(x) = \max_{x \in Q} f(x) - f^*$$

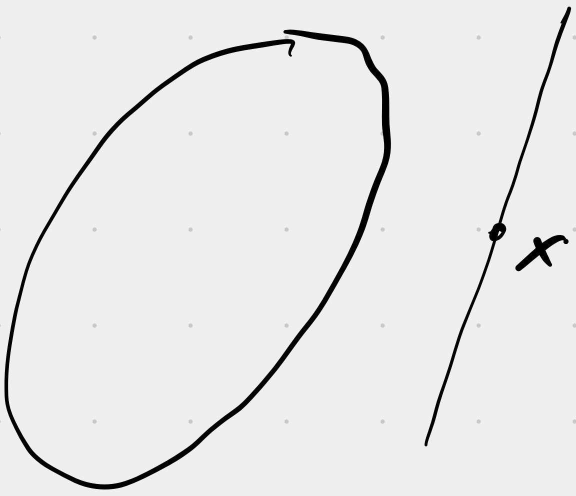
- Parameters: V, R, r, main parameter: n
- The goal: Find $\bar{x} \in Q$: $f(\bar{x}) - f^* \le \varepsilon$.

Separation oracle

We assume we have access:

$$O(x) = \begin{cases} f'(x) \in \partial f(x), & x \in \operatorname{int} Q \\ S_Q(x) \in \mathbb{R}^n, & \text{otherwise} \end{cases}$$

separation hyperplane for Q at $x \notin \operatorname{int} Q$

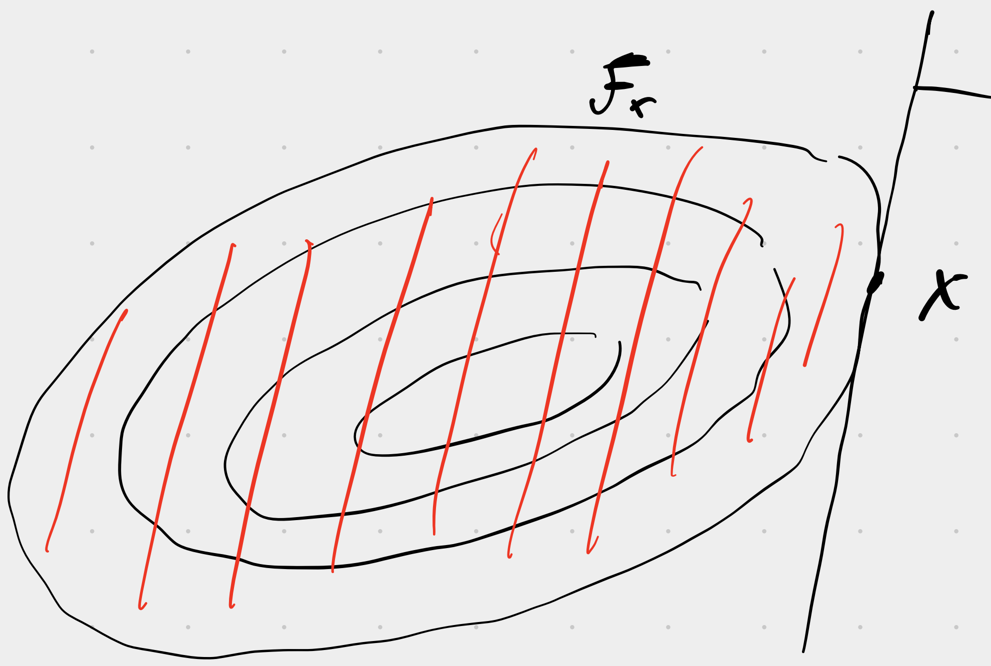


$$\langle S_Q(x), x - y \rangle \geq 0 \quad \forall y \in Q.$$

Otherwise: $x \in \operatorname{int} Q$.

$$\langle f'(x), x - y \rangle \geq f(x) - f(y), \quad \forall y \in Q$$

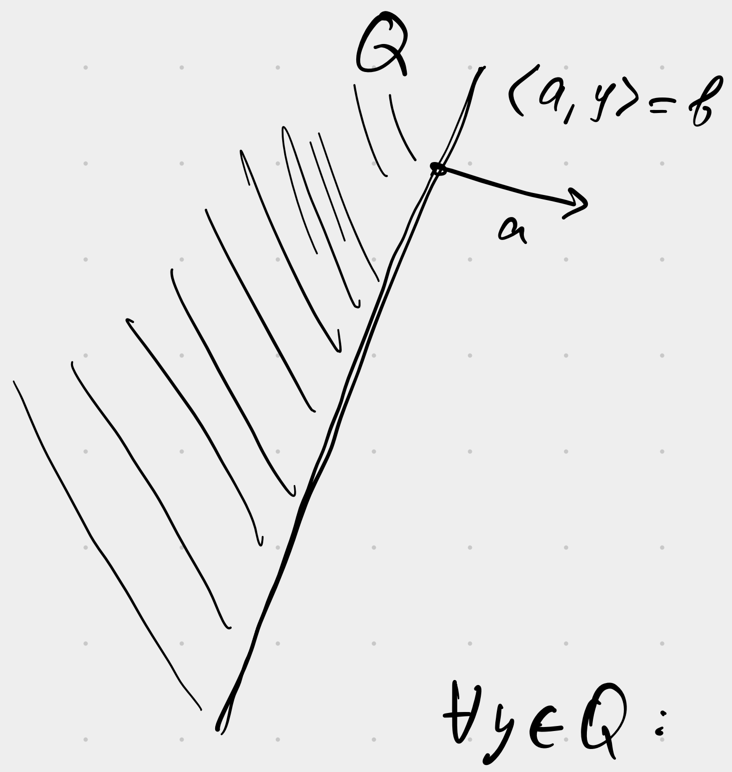
1. $f'(x) = 0 \Rightarrow x$ - global minimum $f(x)$
2. $f'(x) \neq 0$: $x^* \in \{ y \in \mathbb{R}^n \mid \langle f'(x), x - y \rangle \geq 0 \}$



$f'(x)$ - separates x from the sublevel set:

$$F_x = \{ y : f(y) < f(x) \}$$

Example $Q = \{ y \in \mathbb{R}^n \mid \langle a, y \rangle \le b \}, \quad a \in \mathbb{R}^n, b \in \mathbb{R}$



- $x \notin \operatorname{int} Q \Rightarrow \underline{\langle a, x \rangle \ge b} \ge \langle a, y \rangle$

$S_Q(x) = a$

$\forall y \in Q: \langle S_Q(x), x - y \rangle \ge 0.$

Example $Q = \{ y \in \mathbb{R}^n \mid Ay \le b \} =$

$= \{ y \in \mathbb{R}^n \mid a_1^T y \le b_1, \dots, a_m^T y \le b_m \}$

$a_1, \dots, a_m$ - are rows of matrix $A \in \mathbb{R}^{m \times n}$

$S_Q(x) =$ Returns a_i for which the constraint is violated.

$O(m \cdot n)$

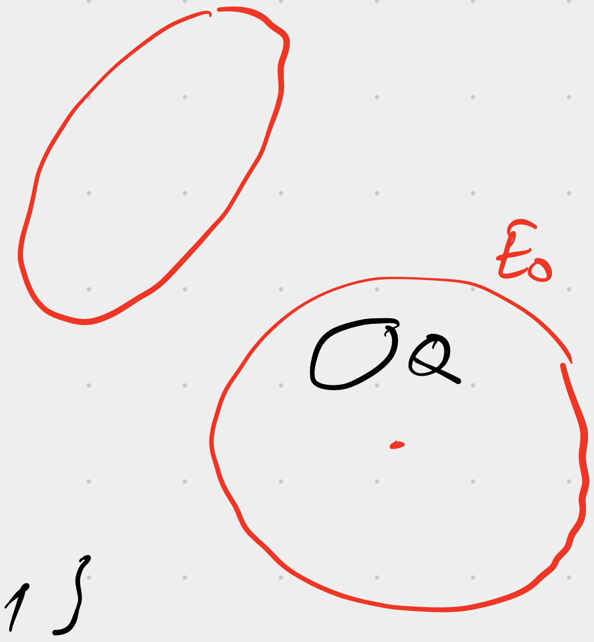
Cutting Plane Scheme

- Search region $G_k = [l_k, r_k]$ - localizer, $x^* \in G_k$
  Binary search
- Next point $x_k \in G_k$ — $x_k = \frac{l_k + r_k}{2}$
- $\text{size}(G_k) \to 0$ — $G_k = r_k - l_k$

in $\mathbb{R}^n$ G_k - "solid" - compact convex set with non-empty interior.

$$G_k = \{ y \in \mathbb{R}^n : \langle g_0, x_0 - y \rangle \geq 0, \dots, \langle g_k, x_k - y \rangle \geq 0 \} \ni x^*$$

Idea: to keep our sets structured
$\downarrow$
Ellipsoids

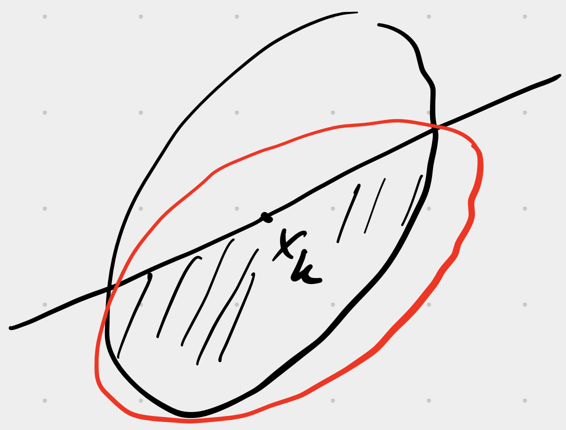


$E_0, E_1, E_2, \dots$ - ellipsoids

$$E_k = \{ y \in \mathbb{R}^n : \langle A_k^{-1}(y - x_k), y - x_k \rangle \leq 1 \}$$

x_k - center of E_k, $A_k = A_k^T \succ 0$.

Goal: Find $E_{k+1} \supseteq \{ y \in E_k \mid \langle g_k, x_k - y \rangle \geq 0 \}$



Notion of Size

$\text{Size}(E_k) \to 0$ $\qquad\qquad$ $\text{size}(K) = (\text{Vol}(K))^{\frac{1}{n}}, \quad K \subseteq \mathbb{R}^n$

1. Monotonicity: $K_1 \subseteq K_2$: $\text{size}(K_1) \leq \text{size}(K_2)$
2. Homogeneity: For $\alpha \geq 0$: $\text{size}(\alpha K) = \alpha\, \text{size}(K)$
3. Translation invariance: $\text{size}(K + x) = \text{size}(K)$

Theorem Consider general cutting plane scheme, $E_0 \supseteq Q$,
For $k \geq 0$:

1. Choose $x_k \in E_k$
2. Access the oracle: $g_k = O(x_k)$
3. $E_{k+1} \supseteq \{ y \in E_k \mid \langle g_k, x_k - y \rangle \geq 0 \}$

Assume for $0 < \delta < 1$: for $k \geq 0$:

$$\text{size}(E_k) \leq \delta\, \text{size}(Q)$$

Then

$$f(\bar{x}_k) - f^* \leq \delta V,$$

$$\bar{x}_k = \text{argmin} \{ f(y) : y \in \{x_0, \ldots, x_k\} \text{ s.t. } y \in \text{int}\, Q \}$$

Proof. Choose $\delta < \gamma \leq 1$. Contraction of our set:

$$Q_\gamma = \gamma Q + (1-\gamma) x^* \subseteq Q$$

$$\text{size}(Q_\gamma) = \gamma\, \text{size}(Q) > \delta\, \text{size}(Q) \geq \text{size}(E_k)$$

$$\Rightarrow Q_\gamma \not\subseteq E_k \quad \exists\, y = \underline{\gamma z + (1-\gamma) x^*} \in Q_\gamma, \quad z \in Q$$

$$\text{s.t. } \underline{y} \notin E_k.$$

$$E_k \supseteq \{ y \in Q : \langle g_0, x_0 - y \rangle \geq 0, \ldots, \langle g_k, x_k - y \rangle \geq 0 \}$$

$$\Rightarrow \quad 0 \leq i \leq k \quad \text{s.t.} \quad \underline{\underline{\langle g_i, x_i - y \rangle < 0}}.$$

$$\text{Since } y \in Q \Rightarrow g_i = f'(x_i), \quad \underline{x_i \in \text{int}\, Q}.$$

$$f(\bar{x}_k) \leq f(x_i) < f(x_i) + \underbrace{\langle f'(x_i), y - x_i \rangle}_{> 0} \leq f(y)$$

$$\leq \gamma f(z) + (1-\gamma) f^*$$

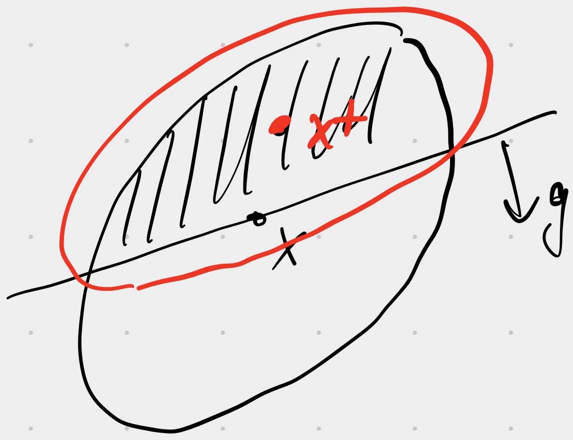
$$f(\bar{x}_k) - f^* \leq \gamma (f(z) - f^*) \leq \gamma V. \qquad \square$$

Ellipsoid Method

Geometric Lemma Let $E \subseteq \mathbb{R}^n$

$$E = \{ y \in \mathbb{R}^n : \langle A^{-1}(y-x), y-x \rangle \le 1 \}$$

Consider $g \in \mathbb{R}^n$



$$x^+ = x - \frac{1}{(n+1)\langle Ag, g\rangle^{1/2}} Ag$$

$$A^+ = \frac{n^2}{n^2-1}\left(A - \frac{2}{(n+1)\langle Ag, g\rangle} A g g^T A\right)$$

1. $E^+ \supseteq \{ y \in E \mid \langle g, x-y \rangle \ge 0 \}$

2. $\mathrm{Vol}(E^+) \le \exp\left(-\frac{1}{2n}\right) \mathrm{Vol}(E)$.

Algorithm: Ellipsoid Method.

Init. $x_0 \in \mathbb{R}^n$, $R > 0$: $B_R(x_0) \supseteq Q$. Set $A_0 = \frac{1}{R} I$

For $k = 0 \ldots K-1$:

1. Access the separation oracle $g_k = O(x_k) \in \mathbb{R}^n$
2. Compute $x_{k+1} = x_k - \frac{1}{(n+1)\langle A_k g_k, g_k\rangle^{1/2}} A_k g_k$ $\quad O(n^2)$
3. $A_{k+1} = \frac{n^2}{n^2-1}\left(A_k - \frac{2}{(n+1)\langle A_k g_k, g_k\rangle} A_k g_k g_k^T A_k\right) \succ 0$

Return $\bar{x}_k$.

Total complexity: $\tilde{O}(n^4)$

Theorem Let $0 < \varepsilon < V$. In order to achieve:

$$f(\bar{x}_k) - f^* \leq \varepsilon$$

we have to do

$$K = \left\lfloor 2n^2 \ln \frac{RV}{r\varepsilon} \right\rfloor + 1$$

separation oracle calls.

Proof $\quad \text{size}(E_k) = \text{Vol}(E_k)^{1/n} \leq \exp\left(-\frac{k}{2n^2}\right) \text{size}(E_0)$

Let $\underline{\delta = \frac{\varepsilon}{V}} < 1$. Then, by main Theorem:

$f(\bar{x}_k) - f^* \leq \varepsilon$ as soon as $\text{size}(E_k) \leq \delta\, \text{size}(Q)$

It sufficient:

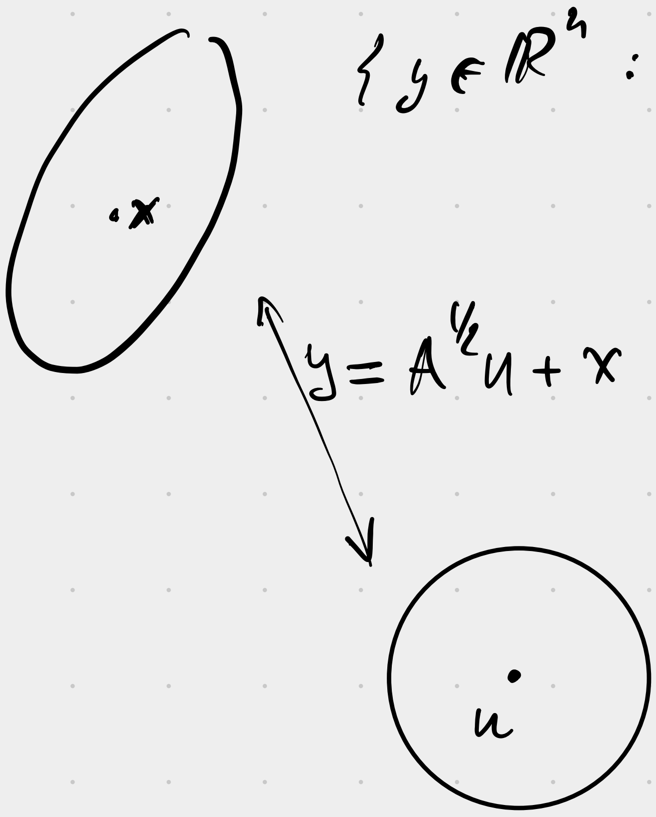
$$\exp\left(-\frac{k}{2n^2}\right) \text{size}(E_0) \leq \delta\, \text{size}(Q)$$

$$\Updownarrow$$

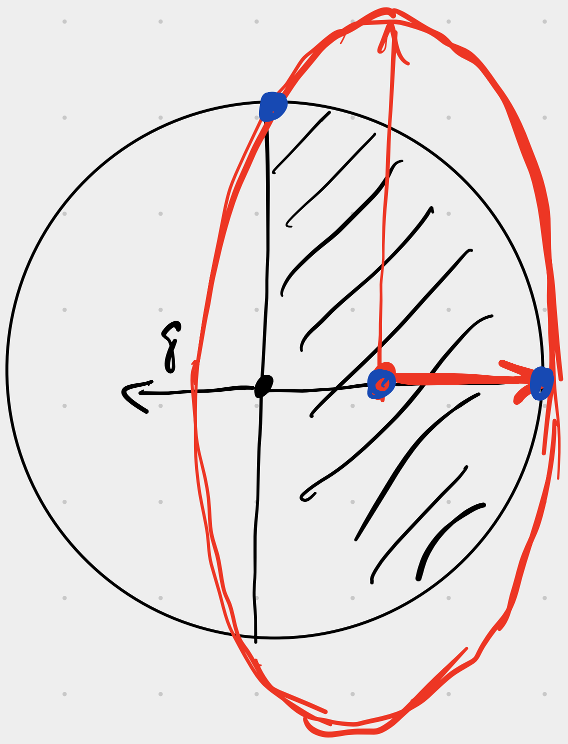
$$k \geq 2n^2 \ln \frac{\text{size}(E_0)}{\delta\, \text{size}(Q)}.$$

$$\frac{\text{size}(E_0)}{\text{size}(Q)} \leq \frac{R}{r}. \quad \square$$

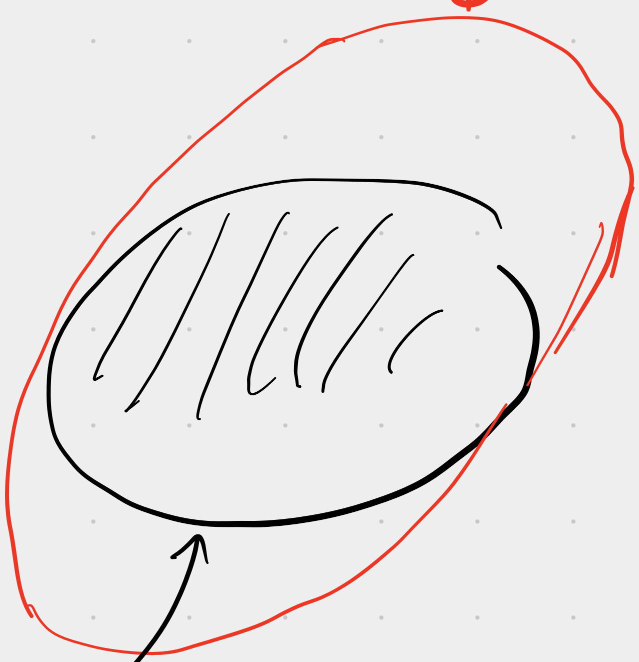
$$\{ y \in \mathbb{R}^n : \langle A^{-1}(y-x), y-x \rangle \leq 1 \}$$



$\|u\|_2 \leq 1$



unique ellipsoid of min volume



any convex solid