Problem

$Q \subset \mathbb{R}^n$ - open convex set, bounded

$$\min\{\langle c,x\rangle,\ x\in \bar{Q}\}$$

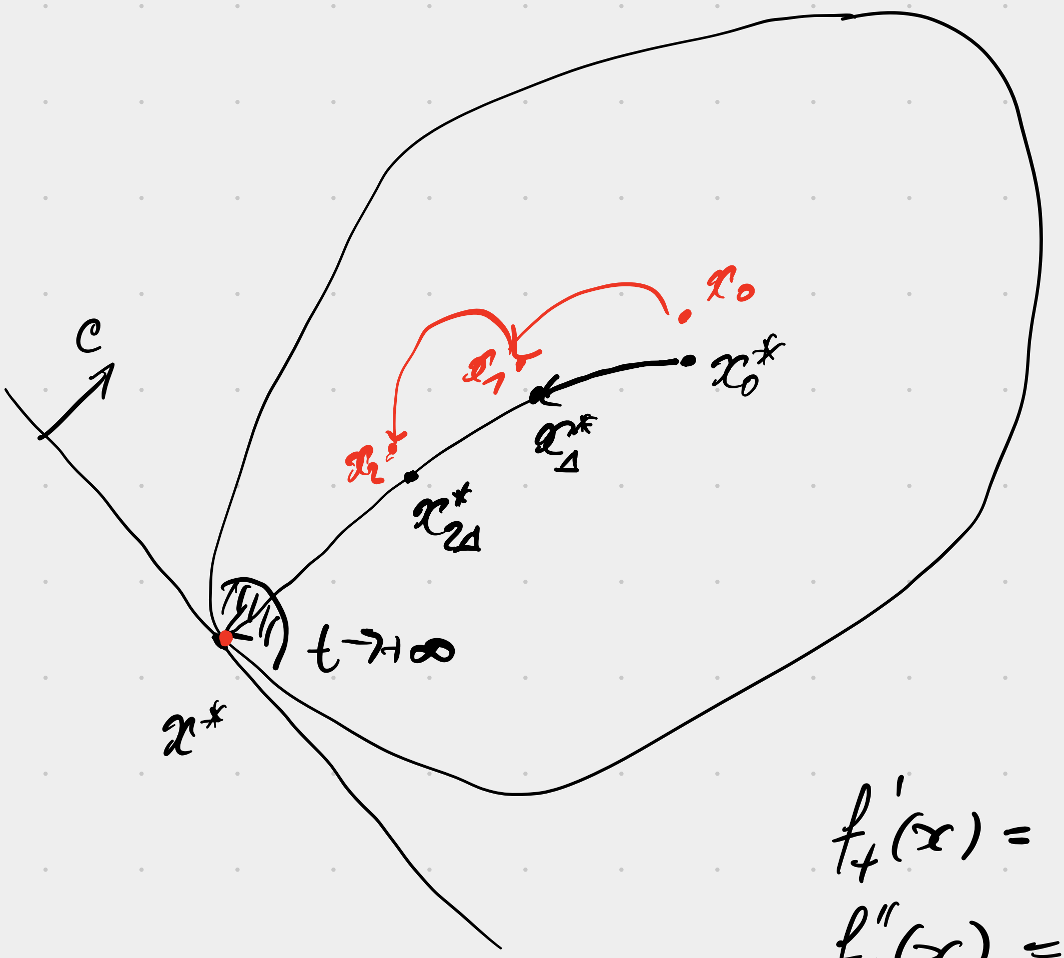
Introduce, $t \geq 0$

$$f_t(x) = t\langle c,x\rangle + F(x)\ ,$$

F - a "barrier" for Q
$\mathrm{dom}\, F = Q$

Denote $\quad x_t^* = \underset{x\in Q}{\mathrm{argmin}}\, f_t(x) = \underset{x\in \mathrm{dom} F}{\mathrm{argmin}}\, f_t(x)$ - central path

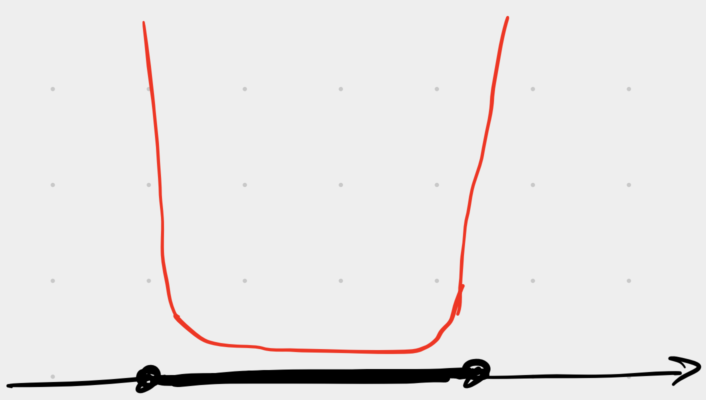
$x_0^* = \mathrm{argmin}\, F(x)$ - an "analytic center" of Q.



$x_t^* \in Q$

Δ - small, not tiny

$$f_t'(x) = tc + F'(x)$$
$$f_t''(x) = \underline{\underline{F''(x)}}$$



Self-Concordant Barriers

Def

We say that $F : Q \to \mathbb{R}$ is a self-concordant barrier for set Q with parameter $\theta > 0$:

1. Q is domain of F: For $x_k \to \partial Q$: $F(x_k) \to +\infty$.
2. F is a standard self-concordant function $(M = 2)$:
$$F'''(x)[h, h, h] \leq 2\|h\|_x^3 \equiv 2\left(F''(x)[h, h]\right)^{3/2}$$
$$\forall x \in Q$$
$$h \in \mathbb{R}^n$$
3. F is Lipschitz w.r.t. local norm:
$$\|DF(x)\|_x^2 = \langle F'(x), F''(x)^{-1} F'(x) \rangle \leq \theta, \quad \forall x \in Q.$$

Fact: $x \in Q$, the entire Dikin's ellipsoid belongs to Q;
$$\mathcal{E}_x = \{ y \in \mathbb{R}^n : \|y - x\|_x < 1 \} \subset Q.$$

Main Result:

$$\boxed{\begin{array}{l}\textbf{Theorem} \text{ to find } \langle c, x_k - x^* \rangle \leq \varepsilon \\ \text{it's enough to perform} \\ \quad K = O\left(\sqrt{\theta} \log \frac{1}{\varepsilon}\right) \\ \text{number of Newton's step.}\end{array}}$$

Equivalent Conditions

1. $\langle F'(x), F''(x)^{-1} F'(x) \rangle \leq \theta \quad \forall x \in Q$

$$\Updownarrow$$

2. $\underbrace{2\langle F'(x), u\rangle - \langle F''(x)u, u\rangle}_{\max u} \leq \theta \quad \forall u \in \mathbb{R}^n, \forall x \in Q$

$F''(x) \succ 0 \iff Q$ does not contain lines

3. Homogenization: $\quad u = \tau h, \quad \tau > 0, \quad h \in \mathbb{R}^n$

$$\max_{\tau > 0} \left[2\tau \langle F'(x), h\rangle - \tau^2 \langle F''(x)h, h\rangle \right] = \frac{\langle F'(x), h\rangle^2}{\langle F''(x)h, h\rangle} \leq \theta$$

For any $h \in \mathbb{R}^n$:

$$\langle F'(x), h\rangle^2 \leq \theta \cdot \langle F''(x)h, h\rangle$$

4. $\forall x \in Q$:

$$F''(x) \succeq \frac{1}{\theta} F'(x) F'(x)^T$$

"self-concordant notion of strong convexity"

Example 1. $F(x) = -\log x$ is a self-concordant barrier, $\boxed{\theta = 1}$

$$F'(x) = -\frac{1}{x}$$

$$F''(x) = \frac{1}{x^2}$$

$$\frac{[F'(x)]^2}{F''(x)} = 1 = \theta$$

Example 2. $F(X) = -\log\det X$ is a self-concordant barrier $\boxed{\theta = n}$

$$DF(X)[H] = -\operatorname{tr}(X^{-1}H) = -\operatorname{tr}(S),$$

$S = X^{-1/2} H X^{-1/2}$ - symmetric

$$D^2F(X)[H,H] = \operatorname{tr}(X^{-1}HX^{-1}H) = \operatorname{tr}(S^2)$$

$$(\operatorname{tr}(S))^2 \overset{?}{\leq} \theta \operatorname{tr}(S^2)$$

$$\left(\sum_{i=1}^n \lambda_i(S)\right)^2 \leq \theta \cdot \sum_{i=1}^n \lambda_i^2(S). \qquad \theta = n$$

$$\|\cdot\|_1 \leq \sqrt{n} \cdot \|\cdot\|_2. \quad \square$$

Basic Operations

Proposition $F_1, \ldots, F_m$ - self-concordant barriers for $Q_1, \ldots, Q_m \subset \mathbb{R}^n$

$\theta_1, \ldots, \theta_m$

Then,

$$Q = \bigcap_{1 \le i \le m} Q_i$$

$$F(x) = \sum_{1 \le i \le m} F_i(x), \quad \theta = \sum_{i=1}^{m} \theta_i .$$

Proof

$$\max_{u \in \mathbb{R}^n} \left[2\langle F'(x), u \rangle - \langle F''(x)u, u \rangle \right] =$$

$$= \max_{u \in \mathbb{R}^n} \left[\sum_{i=1}^{m} \left(2\langle F_i'(x), u \rangle - \langle F_i''(x)u, u \rangle \right) \right]$$

$$\le \sum_{i=1}^{m} \max_{u \in \mathbb{R}^n} \left[- \| - \right] \le \sum_{i=1}^{m} \theta_i = \theta . \qquad \square$$

Example $Q = \{ x \in \mathbb{R}^n \mid x_i > 0,\ 1 \le i \le n \} = \mathbb{R}^n_{>0}$.

$$F(x) = -\sum_{i=1}^{n} \log x_i \qquad \boxed{\theta = n}$$

Affine Invariance $\quad \mathcal{A}(x) = Ax + b$

Let $F: Q \to \mathbb{R}$ - self-concordant barrier for Q, θ_F.

$\Phi(x) = F(\mathcal{A}(x))$ - self-concordant barrier for

$$\Omega = \{ x : \mathcal{A}(x) \in Q \} = \{ x \in \mathcal{A}^{-1}(Q) \}.$$

$$\boxed{\theta_\Phi = \theta_F}$$

Example Linear Programming

$$Q = \{ \langle a_1, x \rangle \le b_1, \dots, \langle a_m, x \rangle \le b_m \}$$

$$F(x) = -\sum_{i=1}^{m} \log(b_i - \langle a_i, x \rangle) \qquad \boxed{\theta = m}$$

$$\left.\begin{array}{l} \min g(x) \\ f(x) \le 0 \end{array}\right\} \to \min_x t g(x) - \log(-f(x)) \qquad \text{"Barrier approach"}$$

CVX, Mosek.

$$\min_x t g(x) + [f(x)]_+^2 \qquad \text{"Penalty Function"}$$

Key Property

lemma $x, y \in Q$

$$\langle F'(x), y-x\rangle \leq \theta.$$

Proof $\varphi(t) = \langle F'(x+t(y-x)), y-x\rangle \quad t \in [0,1]$

$$\varphi(0) \leq \theta$$

$\varphi(0) \leq 0 \Rightarrow$ trivial. Assume $\varphi(0) > 0$.

$$\varphi'(t) = \langle F''(x+t(y-x))(y-x), y-x\rangle$$

$$\geq \frac{1}{\theta}\langle F'(x+t(y-x)), y-x\rangle^2 = \frac{1}{\theta}\varphi(t)^2$$

- $\varphi \nearrow$. $\varphi(t) \geq \varphi(0) \quad \forall\ t \in [0,1]$.

- $\frac{d}{dt}\left[-\frac{1}{\varphi(t)}\right] = \frac{\varphi'(t)}{\varphi(t)^2} \geq \frac{1}{\theta}$

$$\frac{1}{\varphi(0)} \geq \frac{1}{\varphi(0)} - \frac{1}{\varphi(1)} = \int_0^1 \frac{d}{dt}\left[-\frac{1}{\varphi(t)}\right] \geq \frac{1}{\theta} \quad \Rightarrow \quad \varphi(0) \leq \theta. \quad \square$$

Newton's step on Perturbed Problem.

$$x_t^* = \operatorname{argmin}\left[f_t(x) = t\langle c, x\rangle + F(x)\right]$$

Assume $x \approx x_t^*$

$$\|f_t'(x)\|_x = \langle f_t'(x), F''(x)^{-1} f_t'(x)\rangle^{1/2} \leq \delta = \frac{1}{10}.$$

Update time: $t_+ = t + \frac{\gamma}{\|c\|_x}$, $\gamma = \frac{1}{10}$

$$\|c\|_x = \langle c, F''(x)^{-1} c\rangle^{1/2}$$

New objective: $f_{t_+}(y)$

$$x_+ = x - F''(x)^{-1}(t_+ c + F'(x))$$

Theorem

$$\|f_{t^+}'(x^+)\|_{x_+} \leq \delta = \frac{1}{10}.$$