$$\boxed{\begin{array}{l}\min \langle c, x\rangle \\ x \in \bar{Q}\end{array}}$$

$$f_t(x) = t\langle c, x\rangle + F(x),$$

F - self-concord. barrier for Q

update: $t^+ = t + \Delta$

$$f_{t^+}(x) = t^+\langle c, x\rangle + F(x)$$

$$f_{t^+}(x) - f_t(x) = \Delta\langle c, x\rangle$$

$$f'_{t^+}(x) = f'_t(x) + \Delta c$$

$$f''_{t^+}(x) = f'_t(x) = F''(x)$$

Trace the central path?

$$x_t^* = \arg\min_x f_t(x)$$

Proximity to central path:

(x, t): $\quad \|f'_t(x)\|_x \leq \delta = \frac{1}{10}.$

Goal: to find $t^+ = t + \Delta$, x^+

$$\|f'_{t^+}(x^+)\|_{x^+} \leq \delta = \frac{1}{10}$$

$f_{t^+}(x)$ - self-concordant function $\boxed{M = 2}$

Region of local convergence for Newton's method:

$$\lambda(x) \leq \frac{1}{M} \Rightarrow \lambda(x^+) \leq M\lambda(x)^2$$

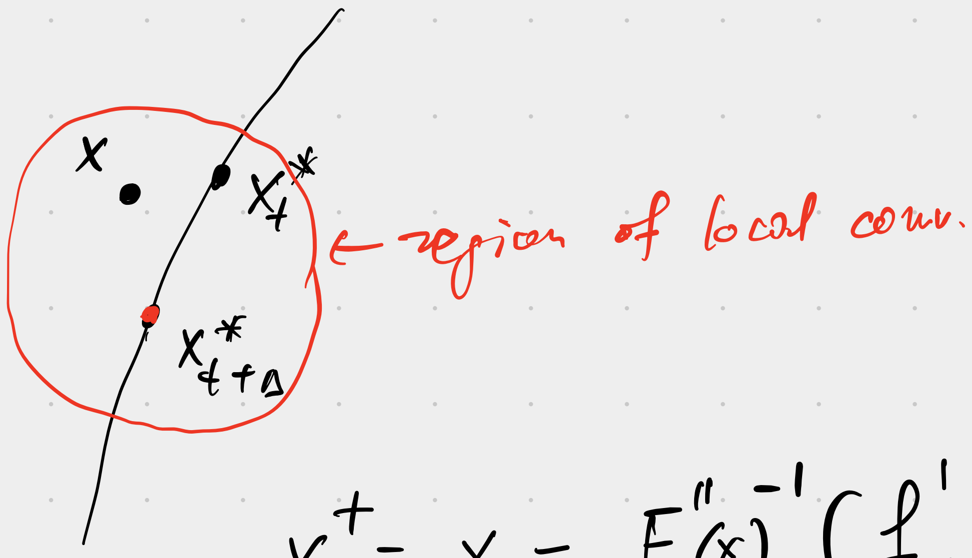
$$\lambda(x) = \langle f'_{t^+}(x), F''(x)^{-1} f'_{t^+}(x)\rangle^{1/2} = \|f'_{t^+}(x)\|_x =$$

$$= \|f'_t(x) + \Delta c\|_x \leq \underbrace{\|f'_t(x)\|_x}_{\leq \delta} + \underbrace{\Delta\|c\|_x}_{\leq \gamma} \leq (\delta + \gamma) = \frac{1}{5}$$

$\Downarrow$

x is in the region of local conv. for $f_{t^+}(x)$

let's $\boxed{\Delta = \frac{\gamma}{\|c\|_x}}$, $\gamma = \frac{1}{10}$



$$x^+ = x - F''(x)^{-1}\left(f'_{t^+}(x)\right)$$

$$\Rightarrow \quad \lambda(x^+) \leq 2\lambda(x)^2 \leq 2(\delta+\gamma)^2 = 2\cdot\frac{1}{25} \leq \frac{1}{10} = \delta$$

$$\lambda(x^+) = \|f'_{t^+}(x^+)\|_{x^+}$$

Theorem let $\delta = \gamma = \frac{1}{10}$. Assume $\|f'_t(x)\|_x \leq \delta$

Update: $t^+ = t + \frac{\gamma}{\|c\|_x} \geq ?$

$$x^+ = x - F''(x)^{-1} f'_{t^+}(x)$$

Then: $\|f'_{t^+}(x^+)\|_{x^+} \leq \delta$.

F-self-concordant barrier:

- $\|F'(x)\|_x^2 \leq \theta$
- Self-limitedness: $\forall x, y \in Q$

  $\langle F'(x), y - x\rangle \leq \theta$

$\Rightarrow \|c\|_x \leq ?$

$$f_t'(x) = tc + F'(x)$$

$$\Rightarrow \quad \|c\|_x = \frac{1}{t}\|f_t'(x) - F'(x)\|_x \leq$$

$$\leq \frac{1}{t}\underbrace{\|f_t'(x)\|_x}_{\leq \delta} + \frac{1}{t}\underbrace{\|F'(x)\|_x}_{\leq \sqrt{\theta}} = \frac{1}{t}\left[\delta + \sqrt{\theta}\right]$$

Lemma $\|f_t'(x)\|_x \leq \delta = \frac{1}{10}$, F - self-concord. Barrier, $\theta > 0$.

Then

① $\|c\|_x \leq \frac{1}{t}\left(\frac{1}{10} + \sqrt{\theta}\right)$

② $t^+ = t + \frac{1}{10 \cdot \|c\|_x} \geq t \cdot \left(1 + \frac{1}{1 + 10\sqrt{\theta}}\right)$.

"linear" increase of time

Convergence rate for central path.

$x_t^* = \text{argmin } t\langle c, x\rangle + F(x) \; : \; f_t'(x_t^*) = tc + F'(x_t^*) = 0$

$$\langle c, x_t^* \rangle - \langle c, x^* \rangle = \langle c, x_t^* - x^* \rangle =$$

$$= \frac{1}{t}\langle F'(x_t^*), x^* - x_t^* \rangle \leq \frac{\theta}{t}.$$

$\Rightarrow$ For central path (exact case): $O\left(\frac{1}{t}\right)$

Inexact: $\|f_t'(x)\|_x \leq \delta = \frac{1}{10}$

$$\Rightarrow \quad \|x - x_t^*\|_x \leq \frac{\delta}{1-\delta} = \frac{1}{9}.$$

$$\langle c, x - x_t^* \rangle \le \|c\|_x \cdot \|x - x_t^*\|_x \le \frac{1}{t}\Big[\underset{\substack{\| \\ \frac{1}{10}.}}{\delta} + \sqrt{\theta}\Big] \cdot \frac{1}{9}$$

$\Rightarrow$

$$\langle c, x \rangle - \langle c, x^* \rangle =$$

$$= \langle c, x_t^* - x^* \rangle + \langle c, x - x_t^* \rangle$$

$$\le \frac{\theta}{t} + \frac{1}{t}\Big[\frac{1}{10} + \sqrt{\theta}\Big] \cdot \frac{1}{9} \underset{(?)}{\le} \varepsilon,$$

At iteration $n \ge 0$:

$$t_{n+1} = t_n + \frac{1}{10\|c\|_{x_n}} \ge t_n\left(1 + \frac{1}{1 + 10\sqrt{\theta}}\right) \ge$$

$$\ge t_1\left(1 + \frac{1}{1 + 10\sqrt{\theta}}\right)^K \underset{(?)}{\ge} \frac{1}{\varepsilon}\Big[\theta + \frac{1}{9}\Big[\frac{1}{10} + \sqrt{\theta}\Big]\Big]$$

It's enough to choose

$$K = O\left(\sqrt{\theta} \ln \frac{1}{t_1 \varepsilon}\right)$$

$$= O\left(\sqrt{\theta} \ln \frac{\|c\|_{x_0}}{\varepsilon}\right)$$

$$t_1 = \frac{1}{10 \cdot \|c\|_{x_0}}$$

Algorithm (Path-Following Interior-Point Method)

Initialization: $\delta = \frac{1}{10}, \gamma = \frac{1}{10}$,

$x_0 \in Q : \|F'(x_0)\|_{x_0} < \delta$.

$t_0 = 0$

Iterate $k \geq 0$:

$$t_{k+1} = t_k + \frac{\gamma}{\|c\|_{x_k}} \qquad \|c\|_{x_k} = \langle c, F''(x_k)^{-1} c \rangle^{1/2}$$

$$x_{k+1} = x_k - F''(x_k)^{-1}\left[F'(x_k) + t_{k+1} c \right]$$

Stop if

$$t_k \geq \frac{1}{9\varepsilon}\left[\sqrt{\theta} + \frac{1}{10}\right] \qquad \text{Return } x_k.$$

$\min\limits_{x \in \bar{Q}} \langle c, x \rangle \Rightarrow$ barrier F for Q.

Theorem To get ε solution:

$$O\left(\sqrt{\theta} \ln \frac{\|c\|_{x_0}}{\varepsilon}\right)$$

How to get to Central Path?

$$x_0^* = \arg\min_{x \in Q} F(x)$$

Auxiliary Path. Assume $y_0 \in Q$

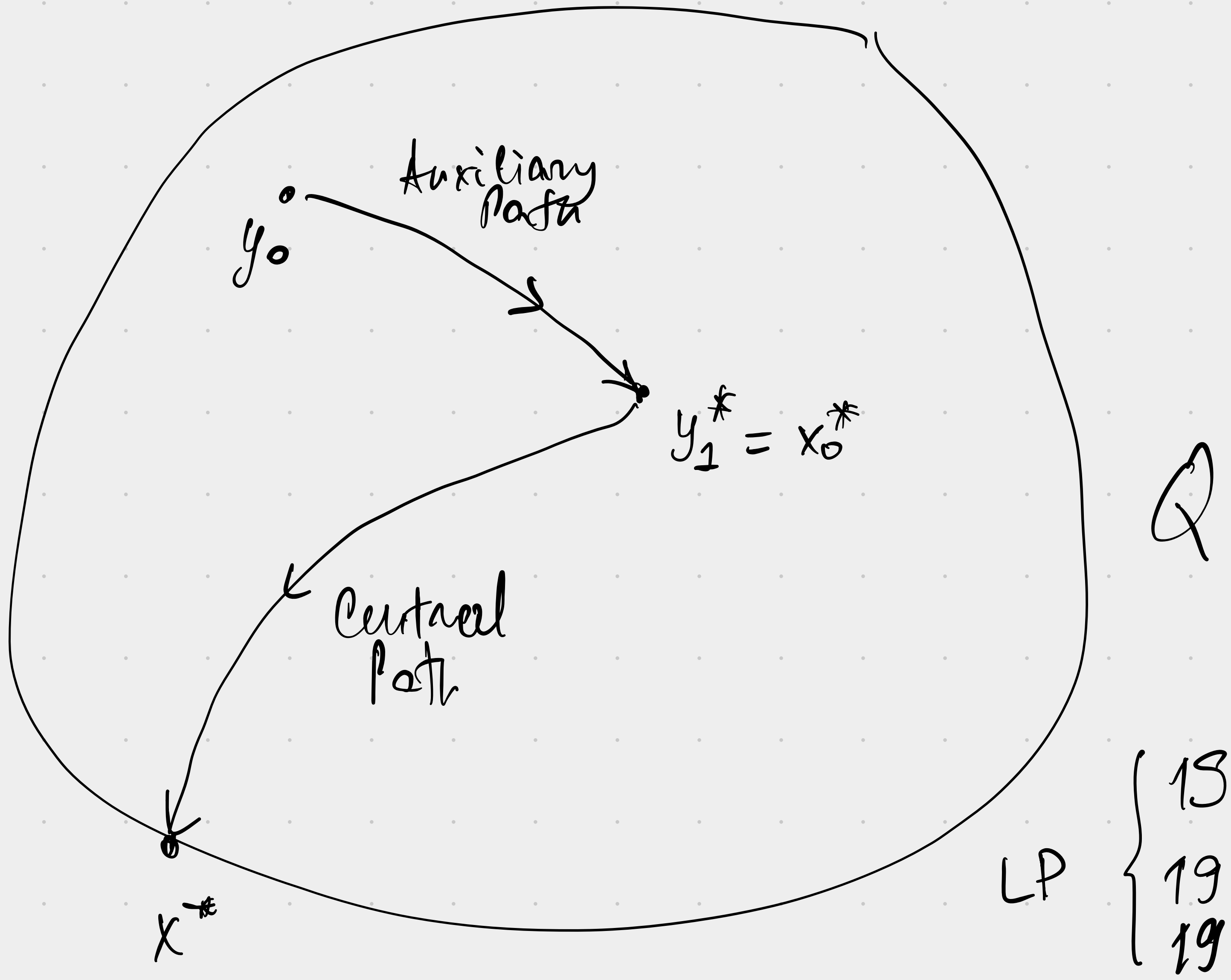
Form $$y_t^* = \arg\min_x \left[-(1-t)\langle F'(y_0), x \rangle + F(x) \right]$$

$$0 \le t \le 1$$

$$-(1-t)F'(y_0) + F'(y_t^*) = 0$$

$t=0$: $F'(y_0) = F'(y_0^*) \Rightarrow \underline{y_0^* = y_0}$

$t=1$: $F'(y_1^*) = 0$



$F''(x) \succ 0$

LP
- 1984 - Karmarkar
- 1986 - Renegar
- 1987 - Gonzaga

SC analysis
- 1994 - Nesterov, Nemirovski

CVXPY

How to find inner point $y_0 \in Q$?

Example: $Q = \{ x : f(x) \le 0 \}$

$$f(x) = \max_{1 \le i \le m} \{ \langle a_i, x \rangle - b_i \}$$

Take $x_0 \in \mathbb{R}^n$: $s_0 = f(x_0) + 1$.

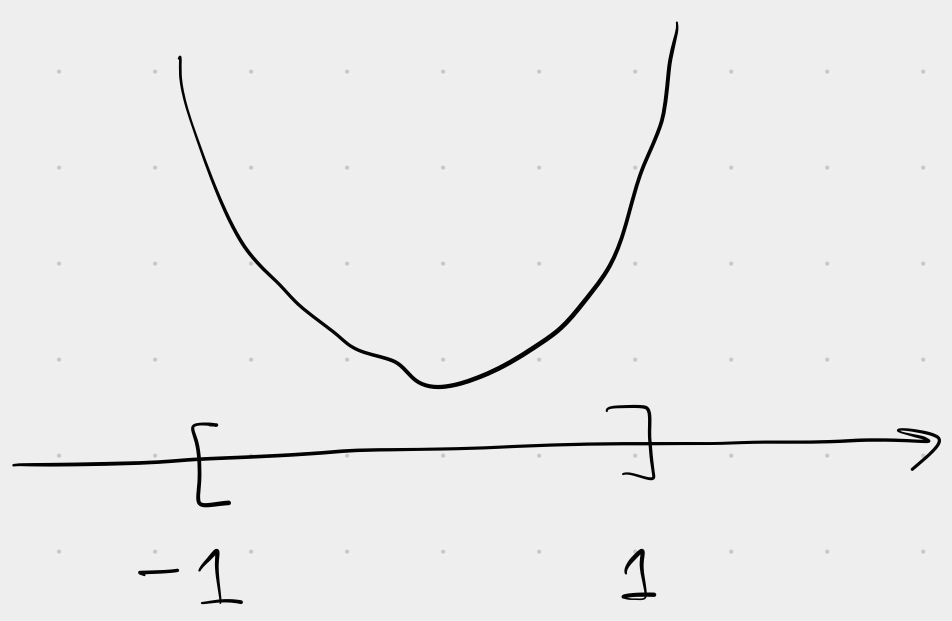
Solve:

$$\boxed{\min_{x, s} \; s \quad \text{s.t.} \quad f(x) \le s}$$

(s_0, x_0) – a feasible inner point.

$Q \subset \mathbb{R}^n \to$ Find a barrier $F(x)$?

Example



$$Q = \{ \underbrace{-1 \le x \le 1}_{} \}$$

$x + 1 \ge 0$

$1 - x \ge 0$

$$F(x) = -\ln(1+x) - \ln(1-x)$$
$$= -\ln(1 - x^2) \quad \Rightarrow \quad \theta = 2$$

$$F(x) = -\ln \cos\left(\frac{\pi}{2} x\right) \quad \Rightarrow \quad \theta = 1$$

Universal Barrier

Polar:

$$P_Q(x) = \{ g \in \mathbb{R}^n : \langle g, y-x \rangle \le 1 \ \forall y \in Q \}$$

Self-limitedness: $\frac{1}{\theta} F'(x) \in P_Q(x)$.

For any set $Q \subset \mathbb{R}^n$:

$$F(x) := c_1 \ln \mathrm{Vol}\, P_Q(x), \qquad c_1 > 0$$

$\Rightarrow$ F self-concordant barrier for Q,

$$\theta = c_2 \cdot n.$$

Complexity of IPM:

$$O\left(\sqrt{n} \log \frac{1}{\varepsilon}\right) \text{ Newton's steps}$$

For center of gravity: $O(n \log \frac{1}{\varepsilon})$

Ellipsoid method: $O(n^2 \log \frac{1}{\varepsilon})$

} separation oracles

$X \succ 0$, $X \in \mathbb{R}^{n \times n}$ $\qquad F(X) = -\ln \det X \Rightarrow \theta = n$

$F'(x), F''(x)$ $\qquad$ (For "universal barrier", $\theta \approx n^2$)