ORIE 6365 — Continuous Optimization: Algorithms and Complexity, **Spring 2026**

Homework 3

Released: April 8, 2026 *Due: April 18, 2026*

Problem 1. Consider the negative entropy distance function,

$$d(x) \;=\; \sum_{i=1}^{n} x^{(i)} \ln x^{(i)}, \qquad x \in \Delta_n,$$

defined on the standard simplex $\Delta_n = \{x \in \mathbb{R}^n_+ \;:\; \langle e, x\rangle = 1\}$, where $e = (1, \ldots, 1)^\top \in \mathbb{R}^n$.

- Show that $x_0 = \left(\frac{1}{n}, \ldots, \frac{1}{n}\right)^\top \in \mathbb{R}^n$ is the minimum of d over Δ_n.
- Consider a step of the mirror descent algorithm:

  $$x_+ \;=\; \arg\min_{y \in \Delta_n} \Big\{ \eta \langle g, y - x\rangle + \beta_d(x; y) \Big\},$$

  where $x \in \Delta_n$ is a current point, $g \in \mathbb{R}^n$ is the direction (the gradient), $\eta > 0$ is a step-size, and

  $$\beta_d(x; y) \;=\; d(y) - d(x) - \langle \nabla d(x), y - x\rangle$$

  is the Bregman divergence associated with d. Prove the following explicit formula for x_+:

  $$x_+^{(i)} \;=\; \frac{x^{(i)} \exp\big(-\eta g^{(i)}\big)}{\sum_{j=1}^{n} x^{(j)} \exp(-\eta g^{(j)})}, \qquad 1 \le i \le n.$$

Problem 2 (bonus).[1] Consider the space of symmetric matrices, $\mathbb{S}^n = \{X \in \mathbb{R}^{n \times n} \;:\; X = X^\top\}$, and the following open *spectrahedron* set, which is a semidefinite generalization of the standard simplex:

$$Q \;=\; \Big\{ X \;\succ\; 0 \;:\; \langle I, X\rangle = 1 \Big\} \;\subset\; \mathbb{S}^n.$$

Recall that for $\mathbb{S}^n$ we use the standard inner product $\langle X, Y\rangle = \mathrm{tr}(XY)$, and I is the identity matrix.

For a symmetric matrix $X \in \mathbb{S}^n$, with a spectral decomposition $X = U\,\mathrm{Diag}(\lambda_1, \ldots, \lambda_n)U^\top$ we define the *matrix exponent* by

$$\exp(X) \;:=\; U\,\mathrm{Diag}(e^{\lambda_1}, \ldots, e^{\lambda_n})U^\top$$

and, when $\lambda_1, \ldots, \lambda_n > 0$ (so $X \succ 0$ is positive definite), the *matrix logarithm* by

$$\ln X \;:=\; U\,\mathrm{Diag}(\ln \lambda_1, \ldots, \ln \lambda_n)U^\top.$$

A natural distance function for spectahedron Q is the Von Neumann entropy:

$$d(X) \;=\; \mathrm{tr}(X \ln X). \tag{1}$$

- Derive the expression for the gradient $\nabla d(X)$ and the Bregman divergence $\beta_d(X; Y)$ associated with distance function (1).
- Compute an explicit formula for the matrix version of the mirror descent:

  $$X^+ \;=\; \arg\min_{Y \in Q} \Big\{ \eta \langle G, Y - X\rangle + \beta_d(X; Y) \Big\}, \qquad X \in Q, \quad G \in \mathbb{S}^n, \quad \eta > 0.$$

[1] This problem is a bonus (extra credit). The core assignment is out of 100 points (Problems 1, 3, 4, and 5). Problem 2 is worth an additional 20 points. Your total score for this homework can exceed 100%, and these extra points will contribute directly to your "Homeworks" category average.

Problem 3. We say that a differentiable convex function $f : Q \to \mathbb{R}$ defined on an open convex set $Q \subseteq \mathbb{R}^n$ is *self-concordant* with constant $M \geq 0$, if

$$D^3 f(x)[h,h,h] \quad \leq \quad M_f \|h\|_x^3, \qquad \forall x \in Q,\ h \in \mathbb{R}^n,$$

where $\|h\|_x := \langle \nabla^2 f(x) h, h \rangle^{1/2}$ is the local norm induced by the Hessian.

- Let $g(y) = f(Ay + b)$, where $A \in \mathbb{R}^{n \times m}$ and $b \in \mathbb{R}^n$. Show that g is self-concordant with the same constant $M_g = M_f$.
- Let $g(x) = cf(x)$, for $c > 0$. What will be the constant of self-concordance M_g for g? Show that for $M_f > 0$, we can always choose c such that $M_g = 2$ (so the function g is "standard self-concordant" after an appropriate rescaling).

Problem 4. We say that a function $f : Q \to \mathbb{R}$ is *quasi-self-concordant* with constant $M \geq 0$, if

$$D^3 f(x)[h,h,u] \quad \leq \quad M_f \|h\|_x^2 \|u\|, \qquad \forall x \in Q,\ h, u \in \mathbb{R}^n,$$

where $\|\cdot\|_x$ is the local norm, and $\|u\| := \langle u, u \rangle^{1/2}$ is the standard Euclidean norm. What will be the constant of quasi-self-concordance M_g after each of the following transformations, as in the previous problem:

- Let $g(y) = f(Ay + b)$ (affine substitution)?
- Let $g(x) = cf(x)$ for $c > 0$ (scaling)?

Problem 5. For each of the following univariate functions, indicate whether its either *self-concordant*, *quasi-self-concordant*, *both*, or *neither*. If yes, show a possible constant $M_f \geq 0$:

- $f(x) = x^4$, $x \in \mathbb{R}$;
- $f(x) = x^4 + x^2$, $x \in \mathbb{R}$;
- $f(x) = e^x$, $x \in \mathbb{R}$;
- $f(x) = \ln(1 + e^x)$, $x \in \mathbb{R}$;
- $f(x) = \frac{1}{x}$, $x > 0$;
- $f(x) = \frac{1}{x} + \frac{x^2}{2}$, $x > 0$.